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Selecting polynomials for the Function Field Sieve

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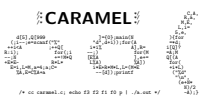

Introduction

- Discrete logarithm is a cryptographic primitive which consists in, for two elements t and s in cyclic group, finding x such that

$$t^x = s.$$

- The Function Field Sieve computes discrete logarithm in $\mathbb{F}_{q^n}$ when q is a small prime power. In our examples $q = 2$. An asymptotically faster algorithm was proposed by Antoine Joux, the crossing point beeing unknown.
- The Caramel team recently announced records in $\mathbb{F}_{2^n}$ for $n = 619$ and 809.
- We study the first step of FFS which selects two polynomials in $\mathbb{F}_q[t][x]$. This choice has consequences on the global time of the computations.

Notations

- We put $F(X, Y) = f(X/Y)Y^{\deg f}$ and we define $G(X, Y)$ similarly.
- For an integer β, a polynomial is β-smooth if it factors into polynomials of degree up to β.
- Example: $f(x) = x^3 + t^2x + 1$, $F(X, Y) = X^3 + t^2XY^2 + Y^3$. For $a = t + 1$ and $b = t^2 + 1$, $F(a, b) = t(t + 1)^3(t^3 + t^2 + 1)$ is 3-smooth.

The Function Field Sieve

Algorithm (Adleman 1994)

Require: a finite field $\mathbb{F}_{q^n}$, two nonzero elements T (a generator) and S
Ensure: $\log_T S$

1: (Polyselect) Select two polynomials f and g in $\mathbb{F}_q[t][x]$ as will be explained;

2: (Sieve) Collect pairs (a, b) in $\mathbb{F}_q[t]$ such that $F(a, b)$ and $G(a, b)$ are smooth;

3: (Linear algebra) Obtain the logarithms of "small" elements by solving a homogeneous system;

4: (Individual logarithm) Express the logarithm of S in terms of the logarithms of the "small" elements.

The polynomial selection method

Algorithm (Joux & Lercier 2002)

Require: a prime power q and a positive integer n

Ensure: two polynomials f and g in $\mathbb{F}_q[t][x]$ and an irreducible polynomial φ in $\mathbb{F}_q[t]$, of degree n, such that f and g have a common root modulo φ

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1: Select a polynomial f with given degree in x and small degree in t;
2: **repeat**
3: $\quad g \leftarrow g_1 x + g_0$ with random g_0, g_1 of given degree
4: **until** $\mathrm{Res}(f, g)$ has a degree n irreducible factor.

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Implication The selection of f is a precomputation. For $\mathbb{F}_{2^n}$ with $n \in [600, 1100]$ one can use:

$$f_{619,1039} = x^6 + (t^2 + t + 1)x^5 + (t^2 + t)x + (t^{12} + t^{10} + t^8 + t^5 + t^3 + t).$$

$$\begin{aligned} g_{619} &= x + t^{104} + 0\text{x6dbb} \\ g_{1039} &= x + t^{174} + 0\text{x1ef9a3} \end{aligned}$$

Candidate polynomials f

- The smoothness probability of $F(a, b)$ and $G(a, b)$ depends on the degree. This imposes the degrees of f and g, e.g. for $\mathbb{F}_2^{809}$ we have $\deg_x f = 6$.
- Sieved coprime pairs (a, b) subject to

$$\begin{aligned} \deg a &\leq e + s/2 \\ \deg b &\leq e - s/2, \end{aligned}$$

where e (sieve param.) and s (skewness) are parameters. We search polynomials f of type

$$f = x^6 + (t^2 + \cdots)x^5 + (t^4 + \cdots)x^4 + \cdots + (t^{12} + \cdots).$$

Problem exposition

We keep the notations previously made.

Problem

Find $f \in \mathbb{F}_q[t][x]$ in order to maximize the number of coprime pairs (a, b) in the sieving domain such that

$$F(a, b) \text{ is } \beta\text{-smooth.}$$

Outline of the talk

- Introduction
- Root property
- Cancellation property
- Conclusion

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Root property: example

Let us now consider the example of $f = x(x-1) - (t^{64} - t)$. For all irreducible ℓ of degree 2, 3 or 6, the reduction of f mod ℓ has two roots. Compare this to a random polynomial having one root modulo each ℓ.

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For any irreducible polynomial ℓ in $\mathbb{F}_q[t]$ we denote by k_ℓ the field $\mathbb{F}_q[t]/\langle \ell \rangle$. Next we put

$$\mathrm{N}(\ell) := \#k_\ell = q^{\deg \ell}.$$

Let ℓ be an irreducible polynomial. The average of $\mathrm{val}_\ell P$ when P is a random polynomial is $\sum_{k \geq 1} \frac{1}{\mathrm{N}(\ell)^k} = \frac{1}{\mathrm{N}(\ell)-1}$.

Alpha

Basic idea Prob($F(a,b)$ smooth)=Prob(P smooth) when $\deg P = \alpha(f) + \deg F(a,b)$.

Definition

For a polynomial f in $\mathbb{F}_q[t][x]$ and an irreducible polynomial ℓ in $\mathbb{F}_q[t]$, we set:

$$\begin{aligned} a_{\text{hom}}^{(\ell)}(f) &= \text{ average of } \operatorname{val}_\ell F(a,b) \text{ when } \gcd(a,b,\ell) = 1; \\ \alpha_\ell(f) &= \deg(\ell)\left(\frac{1}{\mathrm{N}(\ell)-1} - a_{\text{hom}}^{(\ell)}(f)\right); \\ \alpha(f) &= \sum_{\ell \in L} \alpha_\ell(f), \end{aligned}$$

where the last sum will be shown to converge and the average is in the sense of natural density.

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Note that good polynomials have negative values of alpha.

Equation of alpha

Proposition

Let $f \in \mathbb{F}_q[x][t]$ and ℓ a monic irreducible polynomial in $\mathbb{F}_q[t]$ such that ℓ does not divide $\mathrm{Disc}(f)\,\mathrm{Disc}\,F(1,x)$. Then

$$\alpha_\ell(f) = \frac{\deg \ell}{\mathrm{N}(\ell)-1}\left(1 - \frac{\mathrm{N}(\ell)}{\mathrm{N}(\ell)+1} n_\ell\right),$$

where n_ℓ is the number of points $(a : b)$ of $\mathbb{P}^1(k_\ell)$ such that $F(a,b) \equiv 0 \bmod \ell$.

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Proof.

- $$\mathrm{Prob}\left(F(a,b) \equiv 0 \bmod \ell \mid \gcd(a,b,\ell) = 1\right) = \frac{n_\ell}{\mathrm{N}(\ell)+1}.$$

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Proof.

- $$\mathrm{Prob}\left(F(a,b) \equiv 0 \bmod \ell \mid \gcd(a,b,\ell) = 1\right) = \frac{n_\ell}{\mathrm{N}(\ell)+1}.$$
- For all $k \geq 1$, one has

$$\mathrm{Prob}\left(F(\overline{a} + a_k\ell^k, \overline{b} + b_k\ell^k) \equiv 0 \bmod \ell^{k+1} \mid F(\overline{a},\overline{b}) \equiv 0 \bmod \ell^k\right) = \frac{1}{\mathrm{N}(\ell)}.$$

□

Studying alpha

Theorem

Let $f \in \mathbb{F}_q[t][x]$ be absolutely irreducible. Then $\alpha(f)$ is well defined and has a bound depending on $\deg_x f$, $\deg_t f$ and q.

Example

Denote by $\alpha(f, \beta_1)$ the partial sum of f up to degree β_1. Take $q = 2$, and let $f \in \mathbb{F}_q[t][x]$ such that $\deg_x f = 6$ and $g = 19$ and suppose that all the ramified ℓ have degree less than 15. Then one has

$$\begin{aligned} |\alpha(f) - \alpha(f, 15)| &< 0.57 \\ |\alpha(f) - \alpha(f, 20)| &< 0.01. \end{aligned}$$

A sieve procedure for alpha

Instead of considering polynomials f one at a time, we consider pairs ℓ, r and we update the value of $\alpha_\ell(f)$ for all f such that $f_0 \equiv -\sum_{i=1}^{d} f_i r^i \bmod \ell$.

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Example

Paul Zimmermann computed $\sum_{\deg \ell \leq 5} \alpha_\ell(f)$ for 2^{48} polynomials.
The best alpha is -5.5 (while $\alpha(f_{619}) = -4.9$):

$$f_{809} = x^6 + (0x7)x^5 + (0x6b)x^3 + (0x1ab)x^2 + (0x326)x + 0x19b3.$$

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Cancellation property: α_∞

Example

$f = x^3 - t^2x + 1 \in \mathbb{F}_2[t][x]$.
For all (a, b), one has $F(a, b) = a^3 - t^2ab^2 + b^3$.

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- if $\deg(a) = \deg(b) - 2$ then $\deg F(a, b)$ decreases of 1.

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- if $\deg(a) = \deg(b) + 1$ then $\deg F(a, b)$ decreases of 2.

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- if $\deg(a) = \deg(b) - 2$ then $\deg F(a, b)$ decreases of 1.
- if $\deg(a) = \deg(b) + 1$ then $\deg F(a, b)$ decreases of 2.

We call Laurent series (in $1/t$) any expression of the form:

$$r = t^3 + 1/t^2 + 1/t^8 + \cdots .$$

We call $\deg r$, the degree of any of the non-zero truncations.
The non-zero truncations of a Laurent series, together with the degree of their last term are called Laurent polynomials.

Laurent roots

Definition

Let $f \in \mathbb{F}_q[t][x]$ be a polynomial and call d its degree in x. Let (r, m) be a Laurent polynomial. We say that (r, m) is a Laurent root of f if

$$\max_{i \in [0,d]} \deg(f_i r^i) - \deg f(r) > 0. \tag{1}$$

We call gap of (r, m) the least value in the left hand side of the inequality above when we replace r by any Laurent series extending r.

Example

For $f = x^3 + t^2 x + 1$,

$$\begin{aligned} a/b = t + 1/t + O(1/t^2) &\Rightarrow \deg F(a, b) \text{ decreases of } 3 \\ a/b = 1/t^2 + 1/t^8 + O(1/t^9) &\Rightarrow \deg F(a, b) \text{ decreases of } 7. \end{aligned}$$

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Laurent roots correspond to real roots when $f \in \mathbb{Q}[x]$.

Computing Laurent roots

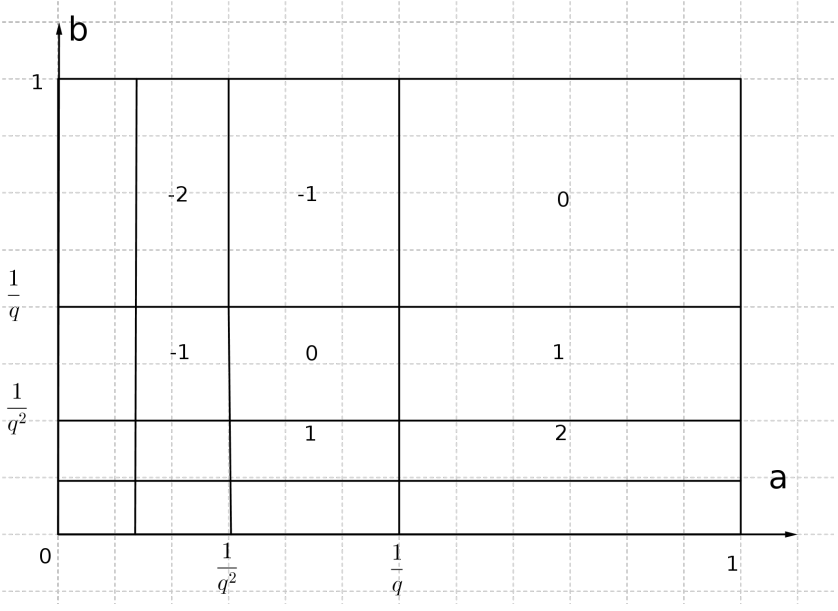
- The first term is found using the integer slopes of the Newton polygon (as for Puiseux series).
- The other terms are computed as in Hensel's lift.
- The gap is computed by trying several expansions of r.

Equation of α_∞

Lemma

For a Laurent root $r + O(\frac{1}{t^{m+1}})$ of $f \in \mathbb{F}_q[t][x]$, and skewness S, one has

$$\text{Prob}\left(\frac{a}{b} = r + O\left(\frac{1}{t^{m+1}}\right)\right) = \frac{q^{-\deg r - m - |s - \deg r|}}{q+1}(1 + O_{e\to\infty}(1/q^{2e})).$$



One obtains a formula for α_∞ similar to α_ℓ, measuring the average loss in degree.

Epsilon

- Recall that we sieve on a domain of skewness s if we impose the condition $\deg a \leq e + s/2$ and $\deg b \leq e - s/2$. We call $\sigma(f, s, e)$ the average degree of $F(a, b)$ if no cancellations occurred.

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- Heuristically, epsilon is the degree of a random polynomial with the same smoothness probability as $F(a, b)$.

Notation

$$\epsilon(f, s, e) = \alpha(f) + \alpha_\infty(f, s) + \sigma(f, s, e),$$

with σ the size of f.

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- Epsilon can be used to estimate the speedup of a polynomial using Dickman's ρ function.

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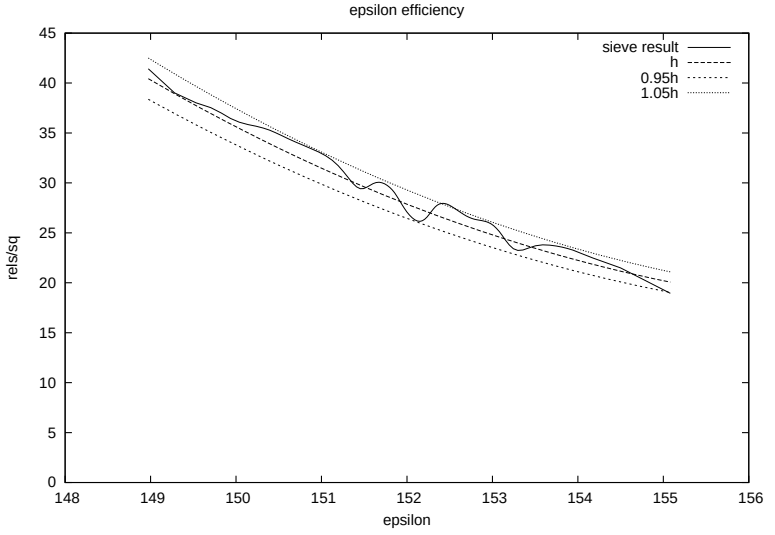
Experimental validation

Figure: Epsilon and sieve efficiency for the polynomials f in our experiment. The function h is a function of type $a + bx + c \log x$, with no special significance.



Conclusion and future work

Conclusion.

- Epsilon compares arbitrary polynomials with a 5% relative error.
- Best polynomials are at least twice as effective as poor ones.
- The results were consistent to the results of the cado-nfs implementation.

The computation of discrete logarithm in $\mathbb{F}_{2^{809}}$.

- Sieve took 17.8k CPU hours and found 117 million pairs (a, b).
- Filtering reduced the number of unknown to 3.68 million.
- Linear algebra, done on GPU, took 13.2 equivalent CPU hours.
- Individual logarithm is negligible and can be run for as many elements as needed.