Real root finding of determinants of linear matrices

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Projet ANR GeoLMI: www.laas.fr/geolmi

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Problem statement

$k \geq 2, n \geq 1$	:	integers
$M_0 \ldots M_n$	:	matrices in $\mathbb{Q}^{k \times k}$

$$\rightarrow \boxed{M(X) = M_0 + X_1 M_1 + \cdots + X_n M_n.}$$

Find points $X \in \mathbb{R}^n$ s.t. rank $M(X) < k$.

$$\updownarrow$$

$$\left\{ X \in \mathbb{R}^n \;\middle|\; \det M(X) = 0 \right\}$$

* A real point in every connected component
* $n = 1$: Real Eigenvalue Problem
* Positive dimensional problem
* Real Roots of $\det(M) = 0$, $\deg(\det(M)) = k$
* First step for solving $\det(M) > 0$ or $\det(M) \geq 0$.

Applications

Stability Analysis of Dynamical Systems

- Hurwitz Criterion: principal minors of Hurwitz matrix $\Delta_1 \geq 0, \ldots, \Delta_s \geq 0$.

Optimisation and Control Theory

- Semidefinite programming (Henrion, Lasserre)

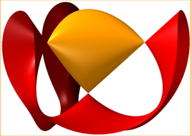
$$\left\{ (X_1, X_2, X_3) \in \mathbb{R}^3 \;\middle|\; \text{eigenvalues of } \begin{bmatrix} 1 & X_1 & X_2 \\ X_1 & 1 & X_3 \\ X_2 & X_3 & 1 \end{bmatrix} \text{ are } \geq 0 \right\}.$$

Geometry of LMI sets:

(a) Points in the boundary (rank ≤ 2)

(b) Singular points = rank 1 matrices:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$



Cayley cubic

State of the art

Existence of real roots

- $F(X_1 \dots X_n) = 0$, $\deg F = k \rightsquigarrow$ complexity $k^{\mathcal{O}(n)}$, hard in practice (*Basu, Pollack, Roy, Grigoriev, Vorobjov, Heintz, Solerno*);
- Using polar varieties (*Bank, Giusti, Heints, Mbakop, Pardo, Safey El Din, Schost*). $\mathcal{O}(k^{3n})$: regular case; $\mathcal{O}(k^{4n})$: singular case.
- Software with Gröbner Bases (`FGb`,`RAGlib`)
- Quadratic case. Complexity: polynomial in n, expon. in the codimension

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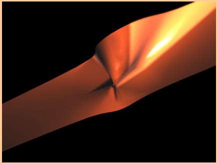
Determinantal structure

- Zero-dimensional case: Gröbner Bases methods $\rightsquigarrow$ complexity bounds (*Faugère, Safey El Din, Spaenlehauer*)
- Positive dimensional case?
- Generic singularity of det M?



Contributions

A new algorithm

- Computing at least one real point in every connected component of a hypersurface defined by a determinant of a linear matrix;
- Complexity:

k	:	size of the matrix
n	:	number of variables

$\rightarrow$ polynomial in $k, n, \binom{k+n}{n}$

- In particular:

k fixed (resp. n) $\rightarrow$ polynomial in n (resp. k)

- implementation in `RAG` : experiments.

Roadmap

Part 1. Modelling: Room-Kempf desingularization

avoid generic singularity $\rightarrow$ structured systems

$\downarrow$

Part 2. Reduction to optimization problems

particular structure $\rightarrow$ dedicated critical points method

$\downarrow$

Part 3. Complexity estimates

multilinear Bézout bounds $\rightarrow$ theoretical and practical improvements

Part 1. Modelling: Room-Kempf desingularization

One polynomial equation : $\det M(X) = 0$

Bilinear system : $\begin{bmatrix} F_1(X,Y) \\ \vdots \\ F_k(X,Y) \end{bmatrix} = M(X) \cdot \begin{bmatrix} Y_1 \\ \vdots \\ Y_k \end{bmatrix} = 0$

$\downarrow$

$V \subset \mathbb{C}^n \times \mathbb{P}_{\mathbb{C}}^{k-1}$

- bilinear in X, Y
- $\dim V = n - 1$, $\deg V = k + \binom{k}{2} + \cdots + \binom{k}{n} \ll 2^k$
- $\Pi_X(V) = \{\det M = 0\}$
- Real Points in V: $(X, Y) \in V \cap \mathbb{R}^{n+k} \to X \in \mathbb{R}^n \cap \{\det M = 0\}$

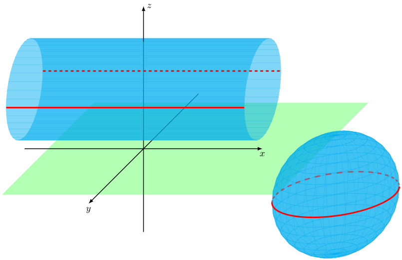
Result: generic smoothness of V.

Part 2. Reduction to optimization problems

The algorithm of Safey El Din / Schost (2003)

Computing real points in every connected component of a smooth variety.

variety → variables gen. position → projections → critical points



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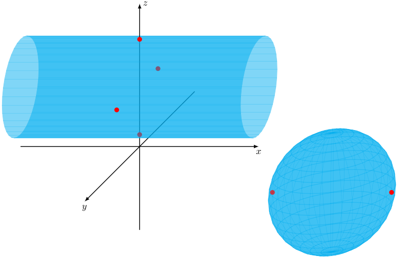

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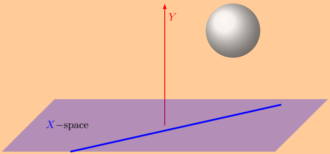
Our case

variety V → gen. position → projections on lines → bilinearity broken

and

multilinear systems → good bounds on the complexity.

- Idea: projections on lines in the X−space:



Part 2. Reduction to optimization problems

One polynomial equation → Bilinear system → **Trilinear system**

$$\boxed{\det M(X) = 0} \rightarrow \boxed{\begin{array}{c} F_1(X, Y) = 0 \\ \vdots \\ F_k(X, Y) = 0 \end{array}} \rightarrow \boxed{\begin{array}{rcl} F_1(X,Y) = \cdots = F_k(X,Y) & = & 0 \\ {}^t Z \cdot \mathrm{Jac}_{\mathbf{X}}(F_1, \ldots, F_k) & = & a \\ {}^t Z \cdot \mathrm{Jac}_{\mathbf{Y}}(F_1, \ldots, F_k) & = & 0. \end{array}}$$

- $a = (a_1 \ldots a_n) \in \mathbb{R}^n$, $\Pi_a(X, Y) = a_1 X_1 + \cdots + a_n X_n$
- Z = Lagrange multipliers
- (X, Y) critical for $\Pi_a\big|V \leftrightarrow \exists\, Z \ : \ (X, Y, Z)$ solution
- $n + 2k - 2$ polynomials, $n + 2k - 2$ variables

Result: generic zero-dimensionality of the Lagrange system.

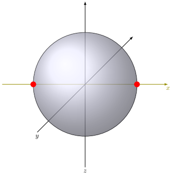
Part 2. Reduction to optimization problems

Geometric idea of the algorithm

* C connected component of $\mathbb{R}^n \cap \{\det M = 0\}$;
* Π_a is generically closed: $\Pi_a(C) = \mathbb{R}$ or $\partial \Pi_a(C) \neq \emptyset$;
* $\Pi_a(X, Y) \in \partial \Pi_a(C) \to X$ critical point (because V is smooth).

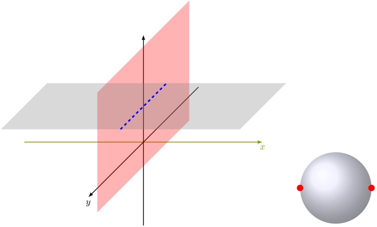
$$\downarrow$$

Either:



$C \cap \Pi_X \left(\mathrm{Crit}(\Pi_a, V)\right) \neq \emptyset$

or:



$\Pi_a^{-1}(0) \cap C \neq \emptyset$

Part 3. Complexity estimates

Geometric resolution

Zero-dimensional system $\longrightarrow$ Univariate representation

- Giusti, Lecerf, Salvy $\to$ complexity $\widetilde{\mathcal{O}}(P(k,n)\cdot(\delta)^2)$.

$$\delta = \max_t \left\{ \deg \left\{ F_1 = \cdots = F_t = 0 \right\} \right\}$$

Multilinear Bezout bounds

k : size of the matrix
n : number of variables

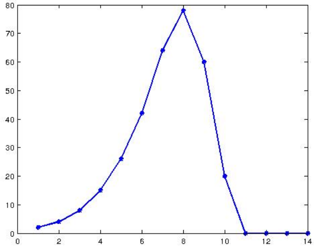
- Part 1: $\delta_1 \leq \cdots \leq \delta_k \leq \binom{k+n}{n}$
- Part 2: $\delta_t = \sum_{\mathcal{F}_t} \binom{k}{i}\binom{t-k}{j}$
- Part 3: $\delta_t = \sum_{\mathcal{F}_t} \binom{k}{i}\binom{n-1}{j}\binom{t-k-n+1}{\ell}$

$$\Longrightarrow \boxed{\delta \leq \binom{\mathbf{k}+\mathbf{n}}{\mathbf{n}}^{2} \ll k^{\mathcal{O}(n)}}$$

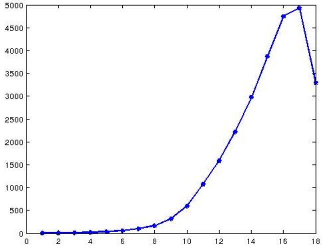
Part 3. Complexity estimates

Asymptotic properties

- $n \gg k$
  - δ_t constant if $n \geq k$ (first and second part)
  - $\delta_t = 0$ if $n \geq 2k$ (third part)
  - maximum: attained in the second part
- $k \gg n$
  - maximum: attained in the third part



$k = 4, n = 8$



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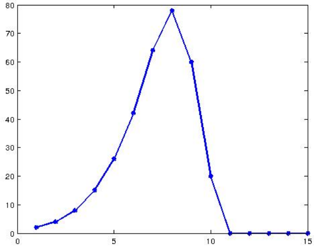
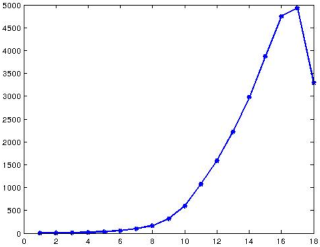


$k = 4, n = 9$



$k = 8, n = 4$

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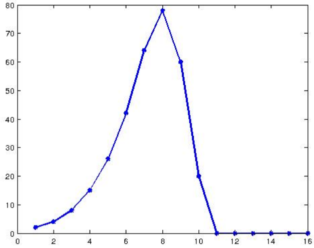
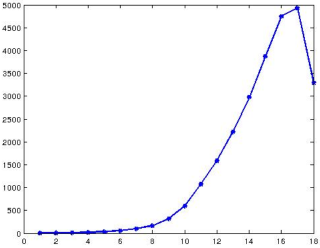


$k = 4, n = 10$



$k = 8, n = 4$

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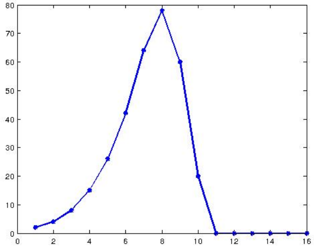
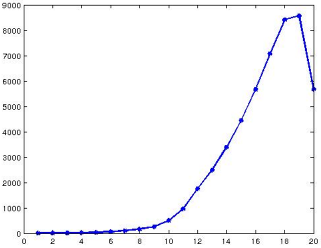


$k = 4, n = 10$



$k = 9, n = 4$

Part 3. Complexity estimates

Asymptotic properties

- $n \gg k$
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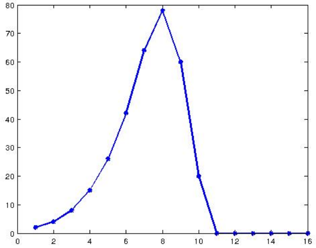
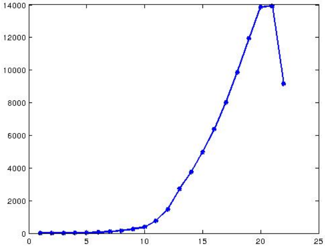
  - maximum: attained in the second part
- $k \gg n$
  - maximum: attained in the third part



$k = 4, n = 10$

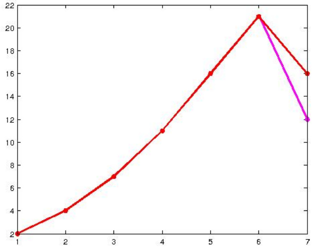


$k = 10, n = 4$

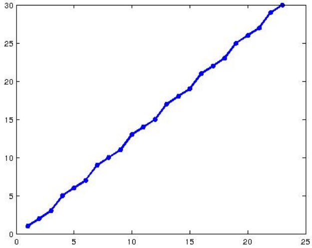
Part 3. Complexity estimates

Experimental conjectures

- the bounds δ_t are sharp for $1 \leq t \leq n + 2k - 3$
- δ_{n+2k-2} is not sharp
- $\delta(k, N(k)) = \max_n(\delta(k, n)) \leq \frac{9^k}{N(k)} \ll \binom{k+N(k)}{k}^2$
- $N(k) = \frac{1}{3}(4k - 1 - (k - 1 \mod 3) \leq \frac{4}{3}k$ for small k.

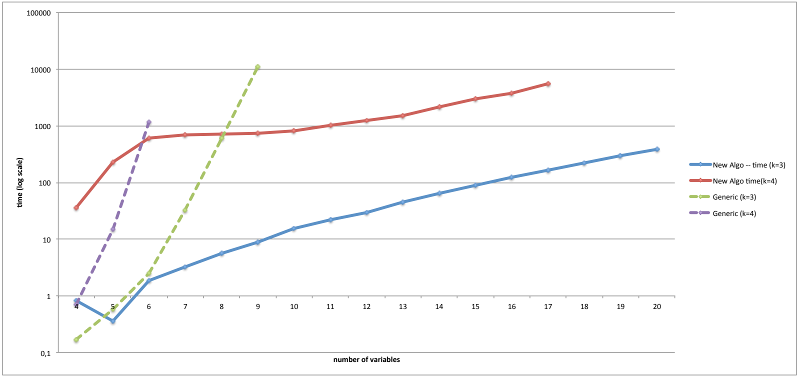


non-sharpness of δ_{n+2k-2}



$N(k)$

Timings



blue: $k = 3, n = 20 \rightsquigarrow$ time $\approx 6,5$ min.
red: $k = 4, n = 17 \rightsquigarrow$ time ≈ 2 h.

Conclusions and outlooks

Conclusions

- an efficient method to solve the problem
- exploiting the geometry of the problem
- a new class of problems solved in polynomial time

Outlooks

- bigger rank drop of linear matrices: $M(X) \cdot Y^1 = \cdots = M(X) \cdot Y^{k-R} = 0$
- positivity problem: $M(X) \geq 0$
- complexity of determinantal representation:

$$P(X) = \det(M_0 + X_1 M_1 + \cdots + X_n M_n).$$

Merci de votre attention!